

Vortex counting from field theory

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Abstract

The vortex partition function in 2d $\mathcal{N} = (2, 2)$ $U(N)$ gauge theory is derived from the field theoretical point of view by using the moduli matrix approach. The character for the tangent space at each moduli space fixed point is written in terms of the moduli matrix, and then the vortex partition function is obtained by applying the localization formula. We find that dealing with the fermionic zero modes is crucial to obtain the vortex partition function with the anti-fundamental and adjoint matters in addition to the fundamental chiral multiplets. The orbifold vortex partition function is also investigated from the field theoretical point of view.

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1 Introduction

The gauge theory partition function plays an essential role in non-perturbative aspects of supersymmetric gauge theory [1, 2]. It is directly given by performing path integral for a certain supersymmetric theory [3], and correctly provides its low energy dynamics [4, 5]. Furthermore it is shown that the instanton partition function is directly interpreted as the conformal block of the two dimensional Liouville field theory [6]. It can be also regarded as a consequence of the M-brane compactifications [7, 8].

Recently partition functions have been provided for the low dimensional gauge theories by performing vortex counting [9, 10, 11, 12, 13], where non-Abelian vortices in U(N) gauge theories [14, 15, 16, 17, 18, 19] play roles of instantons in two dimensions. In addition the moduli space volume itself is also investigated by using the localization formula [20]. They are mainly based on Hanany-Tong's approach [14] in which a D-brane construction is used to describe the vortex moduli space, or on BPS equations themselves. On the other hand, it is shown that, although Hanany-Tong's approach can capture the global structure of the vortex moduli space, the local structure is not correctly treated: the metric of the moduli space is different from the result obtained from the purely field theoretical method [14, 21]. Thus it is important to check its consistency by investigating non-perturbative aspects, e.g. a partition function, from the field theoretical point of view.

Such a non-trivial consequence of the gauge theory is also discussed for the four dimensional orbifold theory, for example, the instanton counting [22], the AGT relation [23, 24, 25, 26, 27, 28], the matrix model description [29, 30] and so on. Recently the low dimensional orbifolds with respect to the vortex moduli space is investigated [31]. The orbifold vortex partition function is given in [32], but it is again based on Hanany-Tong's approach. Thus we should reconsider it in a field theoretical manner as well as the vortex partition functions in the standard two dimensional space.

In this paper we apply the moduli matrix approach to study the vortex moduli space [33, 17, 34, 35, 36, 21, 37]. This is just a purely field theoretical perspective, which can systematically deal with solutions of the BPS equations. The fixed points in the moduli space with respect to the isometry, which is coming from the symmetry of the gauge theory, are completely classified in terms of the moduli matrix. Thus we can write down the character for the tangent space through the moduli matrix method, and then obtain the vortex partition function.

This paper is organized as follows. In section 2, we review the standard derivation of the vortex partition function through the Kähler quotient, on which Hanany-Tong's approach is based. This is parallel to the ADHM construction of instantons, and thus we can obtain the partition function in a quite similar manner to the instanton theory. In section 3, we investigate the vortex partition function from the field theoretical point of view. We discuss the structure of the vortex moduli space for two dimensional $\mathcal{N} = (2, 2)$ $U(N)$ gauge theories, especially the fixed points of the torus action and the corresponding tangent spaces in the moduli space. We write down the character for each tangent space in terms of the moduli matrices, and derive the vortex partition functions for the cases with the anti-fundamental and adjoint matters in addition to $N_F = N$ fundamental chiral multiplets. We find that dealing with the fermionic zero modes is crucial to obtain the vortex partition function with the anti-fundamental and adjoint matters in addition to the fundamental chiral multiplets. In section 4, we extend our result to the orbifold theory on \mathbb{C}/\mathbb{Z}_n . We derive the consistency conditions for the moduli matrices, and then obtain the orbifold partition function by considering the tangent space character at each fixed point. In section 5, we conclude this paper with some remarks and discussions.

2 The Kähler quotient method

The vortex partition functions have been obtained in a similar way to the case of instantons [1] by utilizing the Kähler quotient constructions. In this section, let us review the Kähler quotient method to fix the notation we use in the following discussions.

First, we consider 2d $\mathcal{N} = (2, 2)$ (4d $\mathcal{N} = 1$) $U(N)$ gauge theory with N fundamental chiral multiplets, whose bosonic Lagrangian is given by

$$\mathcal{L}_b = \text{tr} \left[-\mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger - \frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} - \frac{g^2}{4} (HH^\dagger - v^2 \mathbf{1}_N)^2 \right] \quad (2.1)$$

where the adjoint scalar fields in the 2d vector multiplets are omitted because they do not contribute to the BPS equations obtained below. Here H is an N -by- N matrix, on which the color and flavor symmetry act in the following way:

$$H \rightarrow U_C H U_F, \quad U_C \in U(N)_C, \quad U_F \in U(N)_F. \quad (2.2)$$

The BPS vortex equations for this Lagrangian are given by [14, 15]

$$(\mathcal{D}_1 + i\mathcal{D}_2)H = 0, \quad F_{12} + \frac{g^2}{2}(v^2 \mathbf{1}_N - HH^\dagger) = 0. \quad (2.3)$$

The vortex moduli space is given by the space of the BPS solutions with a fixed vortex number k defined by

$$k \equiv -\frac{1}{2\pi} \int_{\mathbb{C}} F. \quad (2.4)$$

A k -vortex solution for this $U(N)$ gauge theory is associated with a pair of matrices (B, I) satisfying [14],

$$[B, B^\dagger] + II^\dagger = r \mathbf{1}_k, \quad (2.5)$$

where r stands for the Fayet-Illiouplos (FI) parameter on the vortex worldvolume.¹ We have $B \in \text{Hom}(V, V)$, $I \in \text{Hom}(W, V)$ for two vector spaces V and W , whose dimensions are interpreted as the winding number and the rank of the gauge group, $\dim V = k$ and $\dim W = N$, respectively. Thus the moduli space is given by

$$\mathcal{M}_{N,k} \cong \left\{ (B, I) \middle| [B, B^\dagger] + II^\dagger = r \mathbf{1}_k \right\} / U(k). \quad (2.6)$$

The $U(k)$ symmetry acts on these data as follows,

$$(B, I) \longrightarrow (gBg^{-1}, gI), \quad g \in U(k). \quad (2.7)$$

Note that there is another representation of the moduli space,

$$\mathcal{M}_{N,k} \cong \{(B, I)\} // \text{GL}(k, \mathbb{C}), \quad (2.8)$$

where the quotient denoted by the double slash $//$ means that points at which the $GL(k, \mathbb{C})$ action is not free should be removed so that the group action is free at any point.

Let us consider the isometry $U(1) \times U(1)^{N-1}$, which acts on the quotient as

$$(B, I) \longrightarrow (T_\epsilon B, IT_a^{-1}), \quad (2.9)$$

where we have denoted the torus action as $T_\epsilon = e^{i\epsilon}$ and $T_a = \text{diag}(e^{ia_1}, \dots, e^{ia_N})$. They are coming from the global symmetry of the system $SO(2) \times SU(N)$: the former is the spatial rotation and the latter is the color-flavor diagonal symmetry. Here a_l satisfy $\sum_{l=1}^N a_l = 0$ and are called the twisted mass parameters. The color-flavor symmetry is broken to $U(1)^{N-1}$ due to this twisted mass parameters.

The fixed points in the vortex moduli space are labeled by an N -tuple of one dimensional partitions, which just consists of N integral entries,

$$\vec{k} = (k_1, \dots, k_N). \quad (2.10)$$

¹This FI parameter is related to the coupling constant of the original $U(N)$ gauge theory as $r = 4\pi/g^2$.

The vortex number is given by $k = k_1 + \dots + k_N$.² We then obtain the character of the vector space at the fixed point specified by \vec{k} as

$$\chi(V) = \sum_{l=1}^N \sum_{i=1}^{k_l} T_{a_l} T_{\epsilon}^{i-1}, \quad \chi(W) = \sum_{l=1}^N T_{a_l}. \quad (2.11)$$

Therefore, the character for the tangent space at the fixed point is given by

$$\begin{aligned} \chi(T_{\vec{k}}\mathcal{M}) &= -(1 - T_{\epsilon}) \chi(V^*) \times \chi(V) + \chi(W^*) \times \chi(V) \\ &= \sum_{l,m}^N \sum_{i=1}^{k_m} T_{a_{ml}} T_{\epsilon}^{-k_l+i-1} = \sum_{l,m}^N \sum_{i=1}^{k_m} e^{ia_{ml}+i(-k_l+i-1)\epsilon}, \end{aligned} \quad (2.12)$$

where we have denoted $a_{ml} = a_m - a_l$. Then, the vortex partition function for $U(N)$ gauge theory with N fundamental chiral multiplets is given by applying the localization formula [39]: it can be found by replacing the sum by the products over the weights as

$$Z_{\vec{k}} = \prod_{l,m}^N \prod_{i=1}^{k_m} \frac{1}{a_{ml} + (-k_l + i - 1)\epsilon}. \quad (2.13)$$

The number of products in this partition function is Nk , which is just the dimension of the moduli space, $\dim_{\mathbb{C}} \mathcal{M}_{N,k} = Nk$.

It is expected that we can also deal with the cases with the anti-fundamental and adjoint matters in a similar manner. In order to apply the method discussed above, we have to first derive the corresponding Kähler quotient to such theories.³ In the following, we show another derivation of the vortex partition function without using the Kähler quotient method. We can directly obtain such a partition function from the field theory.

3 The moduli matrix method

In this section, we show how the partition function (2.13) is derived from the field theoretical viewpoint without using the Kähler quotient. To obtain the partition by applying the localization formula, all we have to do is to derive the character of the torus action at each fixed point in the vortex moduli space. We now explicitly derive this character in terms of the moduli matrix.

3.1 $U(N)$ gauge theory with N fundamental matters

We first consider the bosonic zero modes for the case with N fundamental chiral multiplets. The solution of the BPS equation (2.3) can be written in terms of a holomorphic matrix

² The same decomposition has been made for $SU(N)$ -orbits of the vortex moduli space [38].

³The models with the anti-fundamental and adjoint matters have been so far investigated by using the reduction from the ADHM data for instantons [12, 13].

$H_0(z)$, which we call the moduli matrix. The first equation in (2.3) can be solved as [40, 33, 17]

$$H(z, \bar{z}) = v S^{-1}(z, \bar{z}) H_0(z), \quad A_{\bar{z}} = \frac{1}{2} (A_1 + i A_2) = -i S^{-1}(z, \bar{z}) \partial_{\bar{z}} S(z, \bar{z}), \quad (3.1)$$

where $z = x_1 + i x_2$ is the complex coordinate of the two dimensional space \mathbb{C} . The rank of $H_0(z)$ is N for $U(N)$ gauge theory. Then the second BPS equation can be recast into a equation for $S \in GL(N, \mathbb{C})$, which has a unique solution (up to gauge transformations) for a given $H_0(z)$ [41]. Therefore, all moduli parameters are contained in the moduli matrix $H_0(z)$. This construction is invariant under the so-called “ V -transformation”

$$H_0(z) \rightarrow V(z) H_0(z), \quad S(z, \bar{z}) \rightarrow V(z) S(z, \bar{z}), \quad (3.2)$$

with $V(z) \in GL(N, \mathbb{C})$ being holomorphic with respect to z . Since the original fields $(H, A_{\bar{z}})$ are invariant under this transformation, we can define the following equivalence relation of the moduli matrix:

$$H_0(z) \sim V(z) H_0(z). \quad (3.3)$$

Since the energy (action) of k -vortex configurations is given by

$$T = 2\pi v^2 k = -i \frac{v^2}{2} \oint (dz \partial_z - d\bar{z} \partial_{\bar{z}}) \log |\det H_0(z)|^2, \quad (3.4)$$

the determinant of the moduli matrix for k -vortex solutions takes the form

$$\det H_0(z) = \prod_{i=1}^k (z - z_i). \quad (3.5)$$

Here k is the number of vortices on the complex plane \mathbb{C} and z_i parametrize vortex positions. Thus the vortex moduli space is represented in terms of the moduli matrix as

$$\mathcal{M}_{N,k} \cong \left\{ H_0(z) \mid \det H_0(z) = \mathcal{O}(z^k) \right\} // \{V\text{-transformations}\}. \quad (3.6)$$

Next, let us study the fixed points in the vortex moduli space in order to calculate the character of the torus action. Let $H_0^{\vec{k}}(z)$ be the moduli matrix corresponding to the fixed point with $\vec{k} = (k_1, \dots, k_N)$, which can be represented as

$$H_0^{\vec{k}}(z) = \text{diag}(z^{k_1}, \dots, z^{k_N}). \quad (3.7)$$

By using an appropriate V -transformation, we can see that $H_0^{\vec{k}}(z)$ is invariant under the torus action $T_{\epsilon} = e^{i\epsilon}$ and $T_a = \text{diag}(e^{ia_1}, \dots, e^{ia_N})$,

$$H_0^{\vec{k}}(z) \rightarrow V_{\vec{k}} H_0^{\vec{k}}(T_{\epsilon} z) T_a = H_0^{\vec{k}}(z), \quad (3.8)$$

with the corresponding V -transformation,

$$V_{\vec{k}} = \text{diag}(e^{-i(k_1 \epsilon + a_1)}, \dots, e^{-i(k_N \epsilon + a_N)}). \quad (3.9)$$

In order to study the action of the torus action for the tangent space, let us consider the neighborhood around the fixed point parametrized by a small deviation $\delta H_0(z)$

$$H_0(z) \approx H_0^{\vec{k}}(z) + \delta H_0(z), \quad (3.10)$$

which obeys the following infinitesimal version of the equivalence relation (3.3)

$$\delta H_0(z) \sim \delta H_0(z) + \delta V(z) H_0^{\vec{k}}(z). \quad (3.11)$$

Since $\delta H_0(z)$ is an arbitrary N -by- N matrix whose components are polynomials of z , the vector space of all $\delta H_0(z)$ can be written as

$$\{\delta H_0\} \cong \mathbb{C}^N \otimes \underbrace{(\mathbb{C}[z] \oplus \cdots \oplus \mathbb{C}[z])}_N, \quad (3.12)$$

where $\mathbb{C}[z]$ is the set of polynomials. The vector spaces \mathbb{C}^N and $\mathbb{C}[z] \oplus \cdots \oplus \mathbb{C}[z]$ correspond to the rows and columns of $\delta H_0(z)$, respectively. On the other hand, the space of all infinitesimal V -transformation can be written as

$$\{\delta V H_0^{\vec{k}}\} \cong \mathbb{C}^N \otimes (I_{k_1}[z] \oplus \cdots \oplus I_{k_N}[z]), \quad (3.13)$$

where $I_k[z]$ is the set of polynomials which are multiples of z^k . Therefore, the tangent space is given by

$$\begin{aligned} T_{\vec{k}} \mathcal{M} &\cong \mathbb{C}^N \otimes (\mathbb{C}[z]/I_{k_1}[z] \oplus \cdots \oplus \mathbb{C}[z]/I_{k_N}[z]) \\ &\cong \mathbb{C}^N \otimes (P_{k_1}[z] \oplus \cdots \oplus P_{k_N}[z]), \end{aligned} \quad (3.14)$$

where $P_k[z]$ is the set of polynomials whose degrees are less than k . Since the torus action on δH_0 is written as

$$\delta H_0(z) \rightarrow V_{\vec{k}} \delta H_0(T_{\epsilon} z) T_a, \quad (3.15)$$

the characters of the torus action on \mathbb{C}^N and $P_{k_1}[z] \oplus \cdots \oplus P_{k_N}[z]$ are given by

$$\chi(\mathbb{C}^N) = \text{Tr} [V_{\vec{k}}(z)] = \sum_{l=1}^N (T_{\epsilon}^{k_l} T_{a_l})^{-1}, \quad \chi(P_{k_1}[z] \oplus \cdots \oplus P_{k_N}[z]) = \sum_{l=1}^N T_{a_l} \sum_{i=1}^{k_l} T_{\epsilon}^{i-1}. \quad (3.16)$$

Therefore, the character of the torus action on $T_{\vec{k}} \mathcal{M}$ is

$$\begin{aligned} \chi(T_{\vec{k}} \mathcal{M}) &= \chi(\mathbb{C}^N) \times \chi(P_{k_1}[z] \oplus \cdots \oplus P_{k_N}[z]) \\ &= \sum_{l=1}^N \sum_{m=1}^N \sum_{i=1}^{k_m} T_{a_{ml}} T_{\epsilon}^{-k_l+i-1}. \end{aligned} \quad (3.17)$$

This is consistent with the result from the Kähler quotient shown in (2.12).

Based on the discussion above, we then show an easy way to extract the character from the moduli matrix. Each component of the deviation part of the moduli matrix can be represented as

$$(\delta H_0)_{lm} = \sum_{j=1}^{k_m} c_{lm,j} z^{j-1}. \quad (3.18)$$

Here the number of parameters is the same as the dimension of the moduli space, $\dim_{\mathbb{C}} \mathcal{M}_{N,k} = Nk$. Thus it is natural to interpret them as coordinates of (the tangent space of) the moduli space, which are symbolically denoted as ϕ^i ($i = 1, \dots, Nk$). Indeed the Kähler potential of the moduli space is written down in terms of these parameters of the moduli matrix [36]. The isometry, corresponding to (3.8), acts on the coordinates as

$$c_{lm,j} \longrightarrow e^{i(a_m - a_l) + i(-k_l + j - 1)\epsilon} c_{lm,j}. \quad (3.19)$$

These factors coincide with the contribution to the character of the tangent space (2.12). Actually, we can extract the character of the tangent space from them. Eq. (3.19) shows the torus action on these coordinates $\phi^{i=(l,m,j)} = c_{lm,j}$ is not only linear but also diagonal, that is, the parameters transform as $\phi^i \rightarrow (\mathcal{T}_{\vec{k}})^i_j \phi^j$ with a diagonal matrix $\mathcal{T}_{\vec{k}}$. Its eigenvalues are $e^{i(-k_l + j - 1) + i(a_m - a_l)}$, and thus the character is simply given by

$$\chi(T_{\vec{k}} \mathcal{M}) = \text{Tr} [\mathcal{T}_{\vec{k}}] = \sum_{l,m} \sum_{j=1}^{k_m} e^{i(a_m - a_l) + i(-k_l + j - 1)\epsilon}. \quad (3.20)$$

The trace is taken at the fixed point labeled by the partition \vec{k} . This is consistent with the result from the Kähler quotient shown in (2.12).

Although we have discussed only the bosonic zero modes in this subsection, there also exist fermionic zero modes in the BPS vortex backgrounds. For $1/2$ BPS vortices in $\mathcal{N} = (2, 2)$ theories, there are two conserved supercharges in their effective theories. In the case of $U(N)$ gauge theory with N fundamental chiral multiplets, we can show by examining the equations of motion for the fermions that there is one fermionic moduli parameter ζ_+^i for each bosonic moduli parameter ϕ^i . The parameters (ϕ^i, ζ_+^i) form a supermultiplet under the unbroken supersymmetry and transform in the same way under the torus action. According to the localization formula, if there is a supermultiplet (ϕ^i, ζ_+^i) for which the weight of torus action is λ , its contribution to the vortex partition function is $\lambda^{-2} \times \lambda = \lambda^{-1}$. Here λ^{-2} and λ are the contributions from the bosonic and fermionic part, respectively. Therefore, in the $\mathcal{N} = (2, 2)$ $U(N)$ gauge theory with N fundamental chiral multiplets, the vortex partition function takes the form given in (2.13). In the next two examples, we will see that there exist fermionic zero modes which are not paired with dynamical bosonic zero modes.

3.2 Adding anti-fundamental matters

In this section, we consider vortices in 2d $\mathcal{N} = (2, 2)$ $U(N)$ gauge theory with N fundamental and \tilde{N} anti-fundamental chiral multiplets. In this case, a fermionic version of the moduli

matrix plays an important role. Fermionic zero modes in 4d $\mathcal{N} = 1$ theory in the case of $N = 1$ and $\tilde{N} = 0$ was studied in [42]. The simplest case with anti-fundamental matter, *i.e.*, the case of $N = \tilde{N} = 1$, was studied for a bosonic theory [43] and an $\mathcal{N} = 1$ supersymmetric theory [44].

The relevant part of bosonic Lagrangian is

$$\mathcal{L}_b = \text{tr} \left[\mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger + (\mathcal{D}_\mu \tilde{H})^\dagger \mathcal{D}^\mu \tilde{H} - \frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} - \frac{g^2}{4} (HH^\dagger - \tilde{H}^\dagger \tilde{H} - v^2 \mathbf{1}_N)^2 \right]. \quad (3.21)$$

Here, we have introduced the anti-fundamental field \tilde{H} (\tilde{N} -by- N matrix) in addition to the fields in the Lagrangian (2.1). In this case, the BPS equations are given by

$$0 = \partial_{\bar{z}} H + i A_{\bar{z}} H, \quad (3.22)$$

$$0 = \partial_{\bar{z}} \tilde{H} - i \tilde{H} A_{\bar{z}}, \quad (3.23)$$

$$0 = F_{12} - \frac{g^2}{2} (HH^\dagger - \tilde{H}^\dagger \tilde{H} - v^2 \mathbf{1}_N). \quad (3.24)$$

The general solution of the first two equations are written in terms of the moduli matrix not only for the fundamental field but also for the anti-fundamental field,

$$A_{\bar{z}} = -i S^{-1} \partial_{\bar{z}} S, \quad H = v S^{-1} H_0(z), \quad \tilde{H} = v \tilde{H}_0(z) S. \quad (3.25)$$

Note that both of the moduli matrices, $H_0(z)$ and $\tilde{H}_0(z)$, are holomorphic.

From the fundamental and anti-fundamental fields, the mesonic gauge invariant quantity M can be constructed as

$$M \equiv \tilde{H} H. \quad (3.26)$$

For the BPS configuration, this invariant is a holomorphic function

$$M(z) = v^2 \tilde{H}_0(z) H_0(z). \quad (3.27)$$

Therefore, M must be a constant matrix (otherwise $\lim_{z \rightarrow \infty} M = \infty$). Since $H_0(z)$ must be a rank- N matrix, this condition is satisfied only when $M = 0$, namely

$$\tilde{H}_0(z) = \tilde{H} = 0. \quad (3.28)$$

Thus, only the moduli matrix $H_0(z)$ for the fundamental scalars can be non-trivial, so that the structure of the bosonic part of the moduli space is the same as in the case without the anti-fundamental matters discussed in section 3.1.

Next, let us consider the fermionic part of the theory. Here, we use the convention of 4d $\mathcal{N} = 1$ theories for notational simplicity. The fermionic part of the Lagrangian reads

$$\mathcal{L}_f = i \text{tr} \left[-\frac{1}{g^2} \lambda \sigma^\mu \mathcal{D}_\mu \bar{\lambda} + \bar{\psi} \bar{\sigma}^\mu \mathcal{D}_\mu \psi + \bar{\tilde{\psi}} \bar{\sigma}^\mu \mathcal{D}_\mu \tilde{\psi} - \left\{ \lambda \psi H^\dagger - \tilde{H}^\dagger \tilde{\psi} \lambda - (h.c.) \right\} \right]. \quad (3.29)$$

The equations of motion coming from this part are given by

$$\begin{aligned} 0 &= \bar{\sigma}^\mu \mathcal{D}_\mu \psi + \bar{\lambda} H, \\ 0 &= \bar{\sigma}^\mu \mathcal{D}_\mu \tilde{\psi} + \tilde{H} \bar{\lambda}, \\ 0 &= \sigma^\mu \mathcal{D}_\mu \bar{\lambda} + g^2 (\psi H^\dagger + \tilde{H}^\dagger \tilde{\psi}). \end{aligned} \quad (3.30)$$

These equations can be explicitly written in terms of the Weyl spinors in four dimensions. Let ψ_+, ψ_-, \dots be the components of the Weyl spinors

$$\psi_\alpha = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \tilde{\psi}_\alpha = \begin{pmatrix} \tilde{\psi}_+ \\ \tilde{\psi}_- \end{pmatrix}, \quad \bar{\lambda}^{\dot{\alpha}} = \begin{pmatrix} \lambda_-^\dagger \\ -\lambda_+^\dagger \end{pmatrix}. \quad (3.31)$$

In two dimensions ($\mu = 1, 2$), the covariant derivatives reduce to

$$\sigma^\mu \mathcal{D}_\mu = \begin{pmatrix} 0 & -2\mathcal{D}_z \\ -2\mathcal{D}_{\bar{z}} & 0 \end{pmatrix}, \quad \bar{\sigma}^\mu \mathcal{D}_\mu = \begin{pmatrix} 0 & 2\mathcal{D}_z \\ 2\mathcal{D}_{\bar{z}} & 0 \end{pmatrix}. \quad (3.32)$$

We now solve these equations by introducing fermionic holomorphic functions. In the vortex background, the equations of motion for $\tilde{\psi}$ become⁴

$$0 = \mathcal{D}_{\bar{z}} \tilde{\psi}_+ = \partial_{\bar{z}} \tilde{\psi}_+ - i \tilde{\psi}_+ A_{\bar{z}}, \quad (3.33)$$

$$0 = \mathcal{D}_z \tilde{\psi}_- = \partial_z \tilde{\psi}_- - i \tilde{\psi}_- A_z. \quad (3.34)$$

Thus we have the following general solutions to these equations,

$$\tilde{\psi}_+ = \tilde{\psi}_{0+}(z) S, \quad \tilde{\psi}_- = \tilde{\psi}_{0-}(\bar{z}) S^{\dagger-1}. \quad (3.35)$$

Now let us consider the boundary condition for the fermionic zero modes. We simply assume that the fermionic zero modes vanish at the spatial infinity,

$$\lim_{|z| \rightarrow \infty} \tilde{\psi}_\pm = 0. \quad (3.36)$$

Since the asymptotic form of S for the vortex solution at the fixed point (k_1, \dots, k_N) is (see, e.g., [17])

$$S = \left[1 + \mathcal{O}(e^{-gv|z|}) \right] \text{diag}(|z|^{k_1}, \dots, |z|^{k_N}), \quad (3.37)$$

$\tilde{\psi}_+$ cannot have any regular solution satisfying the boundary condition. Therefore we have

$$\tilde{\psi}_{0+}(z) = 0. \quad (3.38)$$

On the other hand, the general form of the solution for $\tilde{\psi}_-$ yields

$$\tilde{\psi}_{0-}(\bar{z}) = \begin{pmatrix} \sum_{i=1}^{k_1} \zeta_{11,i} \bar{z}^{i-1} & \cdots & \sum_{i=1}^{k_N} \zeta_{1N,i} \bar{z}^{i-1} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{k_1} \zeta_{\tilde{N}1,i} \bar{z}^{i-1} & \cdots & \sum_{i=1}^{k_N} \zeta_{\tilde{N}N,i} \bar{z}^{i-1} \end{pmatrix}, \quad (3.39)$$

⁴ The equations of motion for λ and ψ give the fermionic moduli ζ_+^i with which the bosonic moduli ϕ^i contained in $H_0(z)$ form the supermultiplets (see next subsection for details).

$U(1)_{\mathcal{R}}/U(1)_J$	1	0	-1
0		A_μ	
1	λ_1		λ_2
2		φ	

Table 1: vector multiplet

$U(1)_{\mathcal{R}}/U(1)_J$	1	0	-1
-1		ψ	
0	H		\tilde{H}^\dagger
1		$\tilde{\psi}^\dagger$	

Table 2: hypermultiplet

where $\zeta_{lm,i}$ are Grassmann parameters. Therefore, the fermionic directions in the vortex moduli space are given by

$$\{\tilde{\psi}_{0-}(\bar{z})\} \cong \mathbb{C}^{\tilde{N}} \otimes (P_{k_1}[\bar{z}] \oplus \cdots \oplus P_{k_N}[\bar{z}]). \quad (3.40)$$

The torus action on $\tilde{\psi}_{0-}(\bar{z})$ is defined by

$$\tilde{\psi}_{0-}(\bar{z}) \rightarrow T_\epsilon^{-1} T_m \tilde{\psi}_{0-}(T_\epsilon^{-1} \bar{z}) V_k^\dagger. \quad (3.41)$$

Here $T_m = \text{diag}(e^{im_1}, \dots, e^{im_{\tilde{N}}})$ stands for the torus action corresponding to the twisted masses for the anti-fundamental matters. The overall factor T_ϵ^{-1} is merely a convention and can be absorbed into the twisted masses. Thus the characters of the torus action on $\mathbb{C}^{\tilde{N}}$ and $P_{k_1}[\bar{z}] \oplus \cdots \oplus P_{k_N}[\bar{z}]$ are simply obtained from (3.41),

$$\chi(\mathbb{C}^{\tilde{N}}) = T_\epsilon^{-1} \sum_{f=1}^{\tilde{N}} T_{m_f}, \quad \chi(P_{k_1}[\bar{z}] \oplus \cdots \oplus P_{k_N}[\bar{z}]) = \sum_{l=1}^N \sum_{i=1}^{k_l-1} T_\epsilon^i T_{a_l}. \quad (3.42)$$

Therefore, we have the character for the fermionic tangent space

$$\chi\left(\mathbb{C}^{\tilde{N}} \otimes (P_{k_1}[\bar{z}] \oplus \cdots \oplus P_{k_N}[\bar{z}])\right) = \sum_{f=1}^{\tilde{N}} \sum_{l=1}^N \sum_{i=1}^{k_l-1} T_{m_f} T_\epsilon^{i-1} T_{a_l}. \quad (3.43)$$

This character correctly reproduces the previous result [9, 12, 13]. Note that similarly to the case of the bosonic zero modes discussed in section 3.1, we can easily extract this character (3.43) from the coefficients $\zeta_{lm,i}$ in the fermionic moduli matrix (3.39), which are regarded as the fermionic coordinates of the vortex moduli space. Since fermionic moduli parameters $\zeta_{lm,i}$ are not paired with any bosonic moduli, the contribution to the partition function from the anti-fundamental part is given by

$$Z_k^{\text{antifund}} = \prod_{f=1}^{\tilde{N}} \prod_{l=1}^N \prod_{i=1}^{k_l-1} (m_f + a_l + (i-1)\epsilon). \quad (3.44)$$

3.3 Adding an adjoint matter

Next, let us consider 2d $\mathcal{N} = (2, 2)$ $U(N)$ gauge theory which can be obtained from 2d $\mathcal{N} = (4, 4)$ $U(N)$ gauge theory with N fundamental hypermultiplets by adding a mass term for the adjoint chiral multiplet in the $\mathcal{N} = (4, 4)$ vector multiplet. In this case, we have to deal

with both bosonic and fermionic zero modes similarly to the case with the anti-fundamental chiral multiplets. See [45, 46] for related analysis.

The $\mathcal{N} = (4, 4)$ gauge theory consists of the vector and hypermultiplets, whose field contents (in 4-dimensional notion) and the $U(1)_{\mathcal{R}}$ and $U(1)_J \subset SU(2)_R$ charges are summarized in Table 1 and Table 2. The BPS equations are the same as those in the previous section, that is, Eqs. (3.22)-(3.24) with $\tilde{N} = N$. Therefore, the solution takes the form

$$A_{\bar{z}} = -iS^{-1}\partial_{\bar{z}}S, \quad H = vS^{-1}H_0(z), \quad \tilde{H} = 0. \quad (3.45)$$

The F-term constraint $\varphi H = 0$ implies that we have to choose $\varphi = 0$ since H has the maximal rank in the BPS vortex configurations. As in the previous cases, all the bosonic moduli parameters are contained in the moduli matrix $H_0(z)$.

On the other hand, the equations of motion for the fermions are

$$0 = \bar{\sigma}^\mu \mathcal{D}_\mu \psi + \bar{\lambda}_1 H + \bar{\lambda}_2 \tilde{H}^\dagger + \varphi^\dagger \bar{\tilde{\psi}}, \quad (3.46)$$

$$0 = \bar{\sigma}^\mu \mathcal{D}_\mu \tilde{\psi} + \tilde{H} \bar{\lambda}_1 - H^\dagger \bar{\lambda}_2 - \bar{\psi} \varphi^\dagger, \quad (3.47)$$

$$0 = \sigma^\mu \mathcal{D}_\mu \bar{\lambda}_1 + g^2(\psi H^\dagger + \tilde{H}^\dagger \tilde{\psi}), \quad (3.48)$$

$$0 = \sigma^\mu \mathcal{D}_\mu \bar{\lambda}_2 + g^2(\psi \tilde{H} - H \tilde{\psi}), \quad (3.49)$$

In the BPS background, these equations become

$$\Delta \begin{pmatrix} \psi_+ \\ \frac{i}{g}\lambda_{1+}^\dagger \end{pmatrix} = \Delta \begin{pmatrix} \tilde{\psi}_-^\dagger \\ \frac{i}{g}\lambda_{2-} \end{pmatrix} = 0, \quad \Delta^\dagger \begin{pmatrix} \psi_- \\ \frac{i}{g}\lambda_{1-}^\dagger \end{pmatrix} = \Delta^\dagger \begin{pmatrix} \tilde{\psi}_+^\dagger \\ \frac{i}{g}\lambda_{2+} \end{pmatrix} = 0, \quad (3.50)$$

where Δ and Δ^\dagger are defined by

$$\Delta \equiv \begin{pmatrix} i\mathcal{D}_{\bar{z}}^f & -\frac{g}{2}H_r \\ \frac{g}{2}H_r^\dagger & i\mathcal{D}_z^a \end{pmatrix}, \quad \Delta^\dagger \equiv \begin{pmatrix} i\mathcal{D}_z^f & \frac{g}{2}H_r \\ -\frac{g}{2}H_r^\dagger & i\mathcal{D}_{\bar{z}}^a \end{pmatrix}, \quad (3.51)$$

where the subscript r denotes the fact that H acts as right multiplication and $\mathcal{D}_{\bar{z}}^f$ and \mathcal{D}_z^a are covariant derivatives which act on the fundamental and adjoint fields, respectively. Note that the linearized BPS equations for the bosonic zero modes are given by

$$\Delta \begin{pmatrix} \delta H \\ \delta A_{\bar{z}} \end{pmatrix} = 0. \quad (3.52)$$

The basis of the solutions of the linear differential equation $\Delta\Phi_i = 0$ are given by

$$\Phi_i = \begin{pmatrix} vS^{-1}\frac{\partial}{\partial\phi^i}H_0 \\ 0 \end{pmatrix} + \begin{pmatrix} i\omega_i H \\ -\frac{2}{g}\mathcal{D}_{\bar{z}}\omega_i \end{pmatrix}, \quad \omega_i \equiv -iS^{-1}\left(\Omega\frac{\partial}{\partial\phi^i}\Omega^{-1}\right)S, \quad (3.53)$$

where ϕ^i ($i = 1, \dots, \dim_{\mathbb{C}}\mathcal{M}_{N,k} = Nk$) are bosonic moduli parameters contained in $H_0(z)$. Therefore, the first two equations in (3.50) give the following two fermionic zero modes for

each bosonic moduli parameter ϕ^i :

$$\begin{pmatrix} \psi_+ \\ \frac{i}{g}\lambda_{1+}^\dagger \end{pmatrix} = \zeta_+^i \Phi_i, \quad \begin{pmatrix} \tilde{\psi}_-^\dagger \\ \frac{i}{g}\lambda_{2-} \end{pmatrix} = \zeta_-^i \Phi_i, \quad (3.54)$$

where ζ_\pm^i are fermionic moduli parameters. On the other hand, it has been shown that there is no zero mode for Δ^\dagger [14]. Now, let us consider the transformation property of the moduli parameters $(\phi^i, \zeta_+^i, \zeta_-^i)$ under the torus action. In this case, we have to include the $U(1)_{\mathcal{R}-J}$ symmetry corresponding to the mass term for the adjoint scalar in the vector multiplet. The action of $U(1)_{\mathcal{R}-J}$ on the original fields is given by

$$(H, \psi) \rightarrow e^{-\frac{i}{2}\mathbf{m}}(H, \psi), \quad (\tilde{H}^\dagger, \tilde{\psi}^\dagger) \rightarrow e^{\frac{i}{2}\mathbf{m}}(\tilde{H}^\dagger, \tilde{\psi}^\dagger), \quad (3.55)$$

$$(A_\mu, \lambda_1) \rightarrow (A_\mu, \lambda_1), \quad (\varphi, \lambda_2) \rightarrow e^{i\mathbf{m}}(\varphi, \lambda_2). \quad (3.56)$$

Even in the presence of the adjoint mass \mathbf{m} , the BPS solutions corresponding to the fixed points are not modified, that is, they are classified by $\vec{k} = (k_1, \dots, k_N)$ and specified by the diagonal moduli matrices of the form (3.7). On the other hand, because of the transformation property of H , the matrix $V_{\vec{k}}$ given in (3.9) is modified as

$$V_{\vec{k}} \rightarrow \text{diag}(e^{-i(k_1\epsilon + a_1 + \mathbf{m}/2)}, \dots, e^{-i(k_N\epsilon + a_N + \mathbf{m}/2)}). \quad (3.57)$$

In the case of $\mathbf{m} = 0$, the vortex configurations preserve four supercharges since they are 1/2 BPS states in the $\mathcal{N} = (4, 4)$ theory. The moduli parameters $(\phi^i, \zeta_+^i, \zeta_-^i)$ form supermultiplets (chiral multiplets) in the vortex effective theory and transform in the same way under the torus action. On the other hand, if the adjoint mass \mathbf{m} is non-zero, the multiplets $(\phi^i, \zeta_+^i, \zeta_-^i)$ are decomposed into (ϕ^i, ζ_+^i) and ζ_-^i . We can show from (3.55), (3.56) and (3.57) that the weights of the multiplets (ϕ^i, ζ_+^i) are not modified while the contributions from ζ_-^i are shifted by \mathbf{m} . Therefore, the contribution to the vortex partition function from the fixed point \vec{k} is given by

$$Z_{\vec{k}}^{\text{adj}} = \prod_{l,m}^N \prod_{i=1}^{k_m} \frac{a_{ml} + (-k_l + i - 1)\epsilon + \mathbf{m}}{a_{ml} + (-k_l + i - 1)\epsilon}. \quad (3.58)$$

If we take the mass decoupling limit, $\mathbf{m} \rightarrow \infty$, this reduces to the simple partition function given in (2.13). On the other hand, when we go back to $\mathcal{N} = (4, 4)$ theory by taking the limit of $\mathbf{m} \rightarrow 0$, this contribution becomes trivial. This situation is quite analogous to the four dimensional $\mathcal{N} = 2^*$ theory.

4 Orbifold vortex partition function

We then consider the vortex partition function for the orbifold theory \mathbb{C}/\mathbb{Z}_n [32]. Since supersymmetry on the two dimensional orbifold \mathbb{C}/\mathbb{Z}_n is not preserved [47], availability of

the localization formula is questionable. Thus the partition function we discuss in this section is a quite formal one. Anyway it can be obtained in a similar manner to the orbifold instanton partition function [22]: it is given by considering the invariant sector under the identification of the spatial coordinate as $z \sim \omega z$, where $\omega = \exp(2\pi i/n)$ is the primitive n -th root of unity.

Under the orbifold action $\Gamma = \mathbb{Z}_n$ the torus action behaves as

$$T_\epsilon \longrightarrow \omega T_\epsilon, \quad T_{a_l} \longrightarrow \omega^{p_l} T_{a_l}, \quad T_{m_f} \longrightarrow \omega^{p_f} T_{m_f}, \quad (4.1)$$

where p_l and p_f are parameters, satisfying $0 \leq p_l, p_f \leq n-1$. They characterize the irreducible representation of $\Gamma = \mathbb{Z}_n$ in the flavor space. This is regarded as the twisted boundary condition, a kind of holonomy, for the flavor space. Thus contribution to the character for the fundamental chiral multiplets (2.12) is modified as

$$T_{a_{ml}} T_\epsilon^{-k_l+i-1} \longrightarrow \omega^{p_{ml}-k_l+i-1} T_{a_{ml}} T_\epsilon^{-k_l+i-1}. \quad (4.2)$$

If the extra ω -factor vanishes, it is invariant under the orbifold action $\Gamma = \mathbb{Z}_n$, and thus contributes to the character. This means the Γ -invariant sector is given by

$$p_{ml} - k_l + i - 1 \equiv 0 \pmod{n}. \quad (4.3)$$

The orbifold vortex partition function for N fundamental chiral multiplets is given by a product over the Γ -invariant sector,

$$Z_{\vec{k}, \Gamma} = \prod_{l,m}^N \prod_{\Gamma\text{-inv.}} \frac{1}{a_{ml} + (-k_l + i - 1)\epsilon}. \quad (4.4)$$

The orbifold partition function for the anti-fundamental matter is given in a similar manner. The corresponding contribution to the character (3.43) is modified as

$$T_{m_f} T_\epsilon^{i-1} T_{a_l} \longrightarrow \omega^{p_f+p_l+i-1} T_{m_f} T_\epsilon^{i-1} T_{a_l}. \quad (4.5)$$

The orbifold invariant sector for this part yields

$$p_f + p_l + i - 1 \equiv 0 \pmod{n}. \quad (4.6)$$

Therefore the orbifold partition function for the anti-fundamental matter is given by

$$Z_{\vec{k}, \Gamma}^{\text{antifund}} = \prod_{f=1}^{\tilde{N}} \prod_{l=1}^N \prod_{\Gamma\text{-inv.}} (m_f + a_l + (i-1)\epsilon). \quad (4.7)$$

Here the product is taken over the Γ -invariant sector defined in (4.6). Similarly the orbifold partition function for the adjoint matter theory is also obtained with the same Γ -invariant sector (4.3),

$$Z_{\vec{k}, \Gamma}^{\text{adj}} = \prod_{l,m}^N \prod_{\Gamma\text{-inv.}} \frac{a_{ml} + (-k_l + i - 1)\epsilon + \mathfrak{m}}{a_{ml} + (-k_l + i - 1)\epsilon}. \quad (4.8)$$

This partition function is reduced to (4.4) in the decoupling limit, $\mathfrak{m} \rightarrow \infty$, and becomes trivial in the massless limit, $\mathfrak{m} \rightarrow 0$.

This orbifold partition function is also derived from the moduli matrix method. The moduli matrix approach to the orbifold theory has been investigated in [31], especially in the absence of the twisted mass terms and so on. When we consider the twisted mass terms, the moduli matrix has to satisfy the following condition,

$$H_0(\omega z) = \Omega H_0(z) \tilde{\Omega}^\dagger, \quad (4.9)$$

$$\Omega = \text{diag}(\omega^{k_1+p_1}, \dots, \omega^{k_N+p_N}), \quad \tilde{\Omega} = \text{diag}(\omega^{p_1}, \dots, \omega^{p_N}). \quad (4.10)$$

From both sides of (4.9) the neighborhood around the fixed point yields

$$\begin{aligned} (\delta H_0(\omega z))_{lm} &= \sum_{j=1}^{k_m} \omega^{j-1} c_{lm,j} z^{j-1}, \\ \left(\Omega \delta H_0(z) \tilde{\Omega} \right)_{lm} &= \sum_{j=1}^{k_m} \omega^{k_l+p_l-p_m} c_{lm,j} z^{j-1}, \end{aligned} \quad (4.11)$$

This means the coefficient has to satisfy $c_{lm,j} = 0$ unless $\omega^{j-1} = \omega^{k_l-p_m}$, which is equivalent to the condition (4.3). Thus we can obtain the orbifold partition function in the similar way as the usual case of vortices on \mathbb{C} . We can apply the same argument to the cases with the anti-fundamental and adjoint matters.

5 Summary and discussion

In this paper, we have investigated the vortex partition function from the field theoretical point of view by using the moduli matrix approach. Since the moduli matrix itself is interpreted as the moduli space coordinates, one can easily see how the isometry acts on the tangent space at the fixed points in the vortex moduli space. The corresponding character has been also written in terms of the moduli matrix, and thus we have consistently derived the vortex partition function in a field theoretical way.

We have mainly dealt with 2d $\mathcal{N} = (2, 2)$ $U(N)$ theories with the twisted masses for the chiral multiplets, which break the $SU(N)$ color-flavor diagonal group into the maximal torus $U(1)^{N-1}$. There is a certain variety of the matter contents: we have considered the cases with \tilde{N} anti-fundamental, and the adjoint matters in addition to $N_F = N$ fundamental chiral multiplets. Due to the partial breaking of the supersymmetry, not only the standard bosonic, but also the fermionic moduli matrix also contributes to the vortex partition function.

We have then considered the vortex partition function for the orbifold \mathbb{C}/\mathbb{Z}_n . By studying the consistency conditions for the bosonic and fermionic moduli matrices in the orbifold theory, we have similarly derived the character of the tangent space at the fixed points, and thus the orbifold vortex partition function from the field theoretical perspective. These conditions are regarded as natural extensions of the original one, which is discussed in [31].

We now comment on some possibilities of future works. In this paper, we have studied vortices with additional matters in the anti-fundamental and adjoint representations, in addition to N fundamental matters. When there are more *fundamental* matter fields, $N_F > N$, vortices are called *semi-local* [48, 49]. Non-Abelian semi-local vortices were studied in [50, 51]. Since the vortex counting in the semi-local case was studied for the $U(1)$ gauge theory [10], an extension to non-Abelian semi-local vortices should be explored. In those cases, we can also discuss the relation between the vortex partition functions and the non-perturbative twisted superpotentials which determine the BPS mass spectra in $\mathcal{N} = (2, 2)$ theories [52, 53]. It explains the coincidence of the BPS mass spectra in 2d and 4d gauge theories [54, 55], since semi-local vortices (sigma model instantons) in the non-Abelian vortex world-sheet can be identified with Yang-Mills instantons in the bulk point of view [55, 56, 57]. Along this line, a new 4d/2d correspondence has been recently proposed [58, 59].

A natural extension of the result obtained in this paper is application to other gauge group theories. The moduli matrix approach to the theory with the gauge symmetry of $G \times U(1)$ such as $G = SO, USp$ is investigated in [60, 61, 62]. The merit of field theory approach is that the construction of vortices is available even for arbitrary groups G [63] for which D-brane (Kähler quotient) construction are not known. We can apply the method developed in this paper to such theories, and obtain vortex partition functions. The instanton partition functions are also given for the theories with various gauge group [64, 65], which can be written in terms of the corresponding root systems. It is expected that the vortex partition functions can also be written in a similar manner. It seems also important to study partition functions for quiver gauge theories [13], and their relation to the AGT correspondence [6].

We can consider other two dimensional spaces, e.g. cylinder [66], torus [67, 68], hyperbolic surfaces [69] and more general Riemann surfaces [70, 71, 72, 73]. In particular, we can apply the localization formula to the partition function on S^2 in a similar way to the case of S^4 [74]. Furthermore, it will be interesting to consider various defects which partially preserve the supersymmetry and investigate its relation to the domain-wall partition function [75].

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References

- [1] N. A. Nekrasov, “Seiberg-Witten Prepotential from Instanton Counting,” *Adv. Theor. Math. Phys.* **7** (2004) 831–864, [arXiv:hep-th/0206161](#).

- [2] N. Nekrasov and A. Okounkov, “Seiberg-Witten Theory and Random Partitions,” [arXiv:hep-th/0306238](#).
- [3] G. W. Moore, N. Nekrasov, and S. Shatashvili, “Integrating over Higgs branches,” *Commun. Math. Phys.* **209** (2000) 97–121, [arXiv:hep-th/9712241](#).
- [4] N. Seiberg and E. Witten, “Monopole Condensation, and Confinement in $\mathcal{N} = 2$ Supersymmetric Yang-Mills Theory,” *Nucl. Phys.* **B426** (1994) 19–52, [arXiv:hep-th/9407087](#).
- [5] N. Seiberg and E. Witten, “Monopoles, Duality and Chiral Symmetry Breaking in $\mathcal{N} = 2$ Supersymmetric QCD,” *Nucl. Phys.* **B431** (1994) 484–550, [arXiv:hep-th/9408099](#).
- [6] L. F. Alday, D. Gaiotto, and Y. Tachikawa, “Liouville Correlation Functions from Four-Dimensional Gauge Theories,” *Lett. Math. Phys.* **91** (2010) 167–197, [arXiv:0906.3219 \[hep-th\]](#).
- [7] E. Witten, “Solutions of Four-Dimensional Field Theories via M-theory,” *Nucl. Phys.* **B500** (1997) 3–42, [arXiv:hep-th/9703166](#).
- [8] D. Gaiotto, “ $\mathcal{N} = 2$ Dualities,” [arXiv:0904.2715 \[hep-th\]](#).
- [9] S. Shadchin, “On F-Term Contribution to Effective Action,” *JHEP* **08** (2007) 052, [arXiv:hep-th/0611278](#).
- [10] T. Dimofte, S. Gukov, and L. Hollands, “Vortex Counting and Lagrangian 3-Manifolds,” *Lett. Math. Phys.* **98** (2011) 225–287, [arXiv:1006.0977 \[hep-th\]](#).
- [11] Y. Yoshida, “Localization of Vortex Partition Functions in $\mathcal{N} = (2, 2)$ Super Yang-Mills Theory,” [arXiv:1101.0872 \[hep-th\]](#).
- [12] G. Bonelli, A. Tanzini, and J. Zhao, “Vertices, Vortices & Interacting Surface Operators,” [arXiv:1102.0184 \[hep-th\]](#).
- [13] G. Bonelli, A. Tanzini, and J. Zhao, “The Liouville Side of the Vortex,” *JHEP* **09** (2011) 096, [arXiv:1107.2787 \[hep-th\]](#).
- [14] A. Hanany and D. Tong, “Vortices, Instantons and Branes,” *JHEP* **07** (2003) 037, [arXiv:hep-th/0306150](#).
- [15] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and A. Yung, “NonAbelian superconductors: Vortices and confinement in $\mathcal{N} = 2$ SQCD,” *Nucl. Phys.* **B673** (2003) 187–216, [arXiv:hep-th/0307287](#).

- [16] D. Tong, “TASI lectures on solitons: Instantons, monopoles, vortices and kinks,” [arXiv:hep-th/0509216](#).
- [17] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, “Solitons in the Higgs Phase: the Moduli Matrix Approach,” *J. Phys.* **A39** (2006) R315–R392, [arXiv:hep-th/0602170](#).
- [18] M. Shifman and A. Yung, “Supersymmetric Solitons and How They Help Us Understand Non-Abelian Gauge Theories,” *Rev.Mod.Phys.* **79** (2007) 1139, [arXiv:hep-th/0703267](#).
- [19] D. Tong, “Quantum Vortex Strings: A Review,” *Annals Phys.* **324** (2009) 30–52, [arXiv:0809.5060 \[hep-th\]](#).
- [20] A. Miyake, K. Ohta, and N. Sakai, “Volume of Moduli Space of Vortex Equations and Localization,” *Prog. Theor. Phys.* **126** (2012) 637–680, [arXiv:1105.2087 \[hep-th\]](#).
- [21] T. Fujimori, G. Marmorini, M. Nitta, K. Ohashi, and N. Sakai, “The Moduli Space Metric for Well-Separated Non-Abelian Vortices,” *Phys. Rev.* **D82** (2010) 065005, [arXiv:1002.4580 \[hep-th\]](#).
- [22] F. Fucito, J. F. Morales, and R. Poghossian, “Multi Instanton Calculus on ALE Spaces,” *Nucl. Phys.* **B703** (2004) 518–536, [arXiv:hep-th/0406243](#).
- [23] V. Belavin and B. Feigin, “Super Liouville Conformal Blocks from $\mathcal{N} = 2$ $SU(2)$ Quiver Gauge Theories,” *JHEP* **07** (2011) 079, [arXiv:1105.5800 \[hep-th\]](#).
- [24] T. Nishioka and Y. Tachikawa, “Para-Liouville/Toda Central Charges from M5-Branes,” *Phys. Rev.* **D84** (2011) 046009, [arXiv:1106.1172 \[hep-th\]](#).
- [25] G. Bonelli, K. Maruyoshi, and A. Tanzini, “Instantons on ALE Spaces and Super Liouville Conformal Field Theories,” *JHEP* **08** (2011) 056, [arXiv:1106.2505 \[hep-th\]](#).
- [26] A. Belavin, V. Belavin, and M. Bershtein, “Instantons and 2D Superconformal Field Theory,” *JHEP* **09** (2011) 117, [arXiv:1106.4001 \[hep-th\]](#).
- [27] G. Bonelli, K. Maruyoshi, and A. Tanzini, “Gauge Theories on ALE Space and Super Liouville Correlation Functions,” [arXiv:1107.4609 \[hep-th\]](#).
- [28] N. Wyllard, “Coset Conformal Blocks and $\mathcal{N} = 2$ Gauge Theories,” [arXiv:1109.4264 \[hep-th\]](#).
- [29] T. Kimura, “Matrix Model from $\mathcal{N} = 2$ Orbifold Partition Function,” *JHEP* **09** (2011) 015, [arXiv:1105.6091 \[hep-th\]](#).

- [30] T. Kimura, “ β -Ensembles for Toric Orbifold Partition Function,” *Prog. Theor. Phys.* **127** (2011) 271–285, [arXiv:1109.0004 \[hep-th\]](#).
- [31] T. Kimura and M. Nitta, “Vortices on Orbifolds,” *JHEP* **09** (2011) 118, [arXiv:1108.3563 \[hep-th\]](#).
- [32] J. Zhao, “Orbifold Vortex and Super Liouville Theory,” [arXiv:1111.7095 \[hep-th\]](#).
- [33] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, “Moduli Space of Non-Abelian Vortices,” *Phys. Rev. Lett.* **96** (2006) 161601, [arXiv:hep-th/0511088](#).
- [34] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, “Manifestly Supersymmetric Effective Lagrangians on BPS Solitons,” *Phys. Rev.* **D73** (2006) 125008, [arXiv:hep-th/0602289](#).
- [35] M. Eto, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, and N. Yokoi, “Non-Abelian Vortices of Higher Winding Numbers,” *Phys. Rev.* **D74** (2006) 065021, [arXiv:hep-th/0607070](#).
- [36] M. Eto, K. Hashimoto, G. Marmorini, M. Nitta, K. Ohashi, and W. Vinci, “Universal Reconnection of Non-Abelian Cosmic Strings,” *Phys. Rev. Lett.* **98** (2007) 091602, [arXiv:hep-th/0609214](#).
- [37] M. Eto, T. Fujimori, M. Nitta, K. Ohashi, and N. Sakai, “Dynamics of Non-Abelian Vortices,” *Phys. Rev.* **D84** (2011) 125030, [arXiv:1105.1547 \[hep-th\]](#).
- [38] M. Eto, T. Fujimori, S. Bjarke Gudnason, Y. Jiang, K. Konishi, M. Nitta, and K. Ohashi, “Group Theory of Non-Abelian Vortices,” *JHEP* **11** (2010) 042, [arXiv:1009.4794 \[hep-th\]](#).
- [39] U. Bruzzo, F. Fucito, J. F. Morales, and A. Tanzini, “Multi-Instanton Calculus and Equivariant Cohomology,” *JHEP* **05** (2003) 054, [arXiv:hep-th/0211108](#).
- [40] Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, “All Exact Solutions of a $1/4$ Bogomol’nyi-Prasad-Sommerfield Equation,” *Phys. Rev.* **D71** (2005) 065018, [arXiv:hep-th/0405129](#).
- [41] C.-S. Lin and Y. Yang, “Non-Abelian Multiple Vortices in Supersymmetric Field Theory,” *Commun. Math. Phys.* **304** (2011) 433–457.
- [42] S. C. Davis, A.-C. Davis, and M. Trodden, “ $\mathcal{N}=1$ supersymmetric cosmic strings,” *Phys. Lett.* **B405** (1997) 257–264, [arXiv:hep-ph/9702360 \[hep-ph\]](#).
- [43] A. Penin, V. Rubakov, P. Tinyakov, and S. V. Troitsky, “What becomes of vortices in theories with flat directions,” *Phys. Lett.* **B389** (1996) 13–17, [arXiv:hep-ph/9609257 \[hep-ph\]](#).

- [44] A. Achucarro, A. Davis, M. Pickles, and J. Urrestilla, “Vortices in theories with flat directions,” *Phys.Rev.* **D66** (2002) 105013, [arXiv:hep-th/0109097 \[hep-th\]](#).
- [45] M. Shifman and A. Yung, “Non-Abelian flux tubes in $\mathcal{N}=1$ SQCD: Supersizing world-sheet supersymmetry,” *Phys.Rev.* **D72** (2005) 085017, [arXiv:hep-th/0501211](#).
- [46] M. Edalati and D. Tong, “Heterotic Vortex Strings,” *JHEP* **05** (2007) 005, [arXiv:hep-th/0703045](#).
- [47] A. Adams, J. Polchinski, and E. Silverstein, “Don’t Panic! Closed String Tachyons in ALE Space-Times,” *JHEP* **10** (2001) 029, [arXiv:hep-th/0108075](#).
- [48] T. Vachaspati and A. Achucarro, “Semilocal cosmic strings,” *Phys.Rev.* **D44** (1991) 3067–3071.
- [49] A. Achucarro and T. Vachaspati, “Semilocal and electroweak strings,” *Phys.Rept.* **327** (2000) 347–426, [arXiv:hep-ph/9904229](#).
- [50] M. Shifman and A. Yung, “Non-Abelian semilocal strings in $\mathcal{N}=2$ supersymmetric QCD,” *Phys.Rev.* **D73** (2006) 125012, [arXiv:hep-th/0603134](#).
- [51] M. Eto, J. Evslin, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, and N. Yokoi, “On the moduli space of semilocal strings and lumps,” *Phys.Rev.* **D76** (2007) 105002, [arXiv:0704.2218 \[hep-th\]](#).
- [52] N. Dorey, “The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms,” *JHEP* **9811** (1998) 005, [arXiv:hep-th/9806056](#).
- [53] N. Dorey, T. J. Hollowood, and D. Tong, “The BPS spectra of gauge theories in two-dimensions and four-dimensions,” *JHEP* **9905** (1999) 006, [arXiv:hep-th/9902134](#).
- [54] M. Shifman and A. Yung, “Non-Abelian string junctions as confined monopoles,” *Phys.Rev.* **D70** (2004) 045004, [arXiv:hep-th/0403149](#).
- [55] A. Hanany and D. Tong, “Vortex strings and four-dimensional gauge dynamics,” *JHEP* **04** (2004) 066, [arXiv:hep-th/0403158](#).
- [56] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, “Instantons in the Higgs phase,” *Phys.Rev.* **D72** (2005) 025011, [arXiv:hep-th/0412048](#).
- [57] T. Fujimori, M. Nitta, K. Ohta, N. Sakai, and M. Yamazaki, “Intersecting Solitons, Amoeba and Tropical Geometry,” *Phys.Rev.* **D78** (2008) 105004, [arXiv:0805.1194 \[hep-th\]](#).

- [58] N. Dorey, S. Lee, and T. J. Hollowood, “Quantization of Integrable Systems and a 2d/4d Duality,” *JHEP* **10** (2011) 077, [arXiv:1103.5726 \[hep-th\]](#).
- [59] H.-Y. Chen, N. Dorey, T. J. Hollowood, and S. Lee, “A New 2d/4d Duality via Integrability,” *JHEP* **09** (2011) 040, [arXiv:1104.3021 \[hep-th\]](#).
- [60] M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, T. Nagashima, M. Nitta, K. Ohashi, and W. Vinci, “Non-Abelian Vortices in $SO(N)$ and $USp(N)$ Gauge Theories,” *JHEP* **06** (2009) 004, [arXiv:0903.4471 \[hep-th\]](#).
- [61] M. Eto, T. Fujimori, S. B. Gudnason, M. Nitta, and K. Ohashi, “SO and USp Kähler and Hyper-Kähler Quotients and Lumps,” *Nucl.Phys.* **B815** (2009) 495–538, [arXiv:0809.2014 \[hep-th\]](#).
- [62] M. Eto, T. Fujimori, S. B. Gudnason, Y. Jiang, K. Konishi, M. Nitta, and K. Ohashi, “Vortices and Monopoles in Mass-deformed SO and USp Gauge Theories,” *JHEP* **12** (2011) 017, [arXiv:1108.6124 \[hep-th\]](#).
- [63] M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, M. Nitta, K. Ohashi, and W. Vinci, “Constructing Non-Abelian Vortices with Arbitrary Gauge Groups,” *Phys.Lett.* **B669** (2008) 98–101, [arXiv:0802.1020 \[hep-th\]](#).
- [64] N. Nekrasov and S. Shadchin, “ABCD of Instantons,” *Commun. Math. Phys.* **252** (2004) 359–391, [arXiv:hep-th/0404225](#).
- [65] C. A. Keller, N. Mekareeya, J. Song, and Y. Tachikawa, “The ABCDEFG of Instantons and W-Algebras,” *JHEP* **03** (2012) 045, [arXiv:1111.5624 \[hep-th\]](#).
- [66] M. Eto, T. Fujimori, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, and N. Sakai, “Non-Abelian Vortices on Cylinder: Duality Between Vortices and Walls,” *Phys. Rev.* **D73** (2006) 085008, [arXiv:hep-th/0601181](#).
- [67] M. Eto, T. Fujimori, M. Nitta, K. Ohashi, K. Ohta, and N. Sakai, “Statistical Mechanics of Vortices from D-Branes and T- Duality,” *Nucl. Phys.* **B788** (2008) 120–136, [arXiv:hep-th/0703197](#).
- [68] G. S. Lozano, D. Marques, and F. A. Schaposnik, “Non-Abelian Vortices on the Torus,” *JHEP* **09** (2007) 095, [arXiv:0708.2386 \[hep-th\]](#).
- [69] N. S. Manton and N. Sakai, “Maximally Non-Abelian Vortices from Self-Dual Yang–Mills Fields,” *Phys. Lett.* **B687** (2010) 395–399, [arXiv:1001.5236 \[hep-th\]](#).
- [70] A. D. Popov, “Non-Abelian Vortices on Riemann Surfaces: an Integrable Case,” *Lett. Math. Phys.* **84** (2008) 139–148, [arXiv:0801.0808 \[hep-th\]](#).

- [71] J. M. Baptista, “Non-Abelian Vortices on Compact Riemann Surfaces,”
Commun. Math. Phys. **291** (2009) 799–812, [arXiv:0810.3220 \[hep-th\]](#).
- [72] J. M. Baptista, “On the L^2 -Metric of Vortex Moduli Spaces,”
Nucl. Phys. **B844** (2011) 308–333, [arXiv:1003.1296 \[hep-th\]](#).
- [73] N. S. Manton and N. A. Rink, “Geometry and Energy of Non-Abelian Vortices,”
J. Math. Phys. **52** (2011) 043511, [arXiv:1012.3014 \[hep-th\]](#).
- [74] V. Pestun, “Localization of Gauge Theory on a Four-Sphere and Supersymmetric Wilson Loops,” [arXiv:0712.2824 \[hep-th\]](#).
- [75] K. Ohta, “Counting BPS Solitons and Applications,” [arXiv:0710.4011 \[hep-th\]](#).